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Hilfer分数阶脉冲随机发展方程的平均原理*

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摘　要： 利用分数阶微积分理论、半群性质、不等式技巧和随机分析理论，建立了分数布朗运动驱动的Hilfer分数阶脉冲随机发展方程的平均原理，证明了原方程的适度解均方收敛于无脉冲平均方程的适度解，并通过实例说明了所得理论结果的适用性.

关键词： 平均原理；Hilfer分数阶导数；脉冲随机发展方程；分数布朗运动

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Averaging principle for Hilfer fractional impulsive stochastic evolution equations

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Abstract: By using fractional calculus, semigroup theories, inequality techniques and stochastic analysis theories, an averaging principle for Hilfer fractional impulsive stochastic evolution equations driven by fractional Brownian motion is established. The mild solution of the original equations converges to the mild solution of the reduced averaged equations without impulses in the mean square sense is proved. And an example is presented to illustrate the applicability of our obtained theoretical results.

Key words: averaging principle; Hilfer fractional derivative; impulsive stochastic evolution equations; fractional Brownian motion

在实际生活中，系统常受外力影响或内部产生的“噪声”干扰，所以，随机微分方程可以更加准确的刻画系统的变化特征，因而研究随机微分方程是很有必要的且存在实际的应用价值. 另外，现实生活中的许多现象都有长期后效作用，Mandelbrot et al.(1968)研究表明分数布朗运动可以较好的描述长期后效现象，这推动了更多学者们对分数布朗运动驱动的随机微分方程的广泛关注. 分数布朗运动(fBm)最早是由Kolmogorov(1940)提出的一个依赖于Hurst参数$H\in(0,1)$的高斯随机过程，当$H=1/2$时，分数布朗运动简化为标准布朗运动；当$H\neq 1/2$时，分数布朗运动既不是半鞅也不是Markov过程；当$H>1/2$时，分数布朗运动具有自相似性、长时记忆性等特征，这些性质使分数布朗运动可以引入到数理金融(Bollerslev et al., 1996)、网络通信(Leland et al., 1994)、生物医学工程(de la Fuente et al., 2006; Boudrahem et al., 2009)等随机模型中作为随机噪声项，得以更好的描述系统特征和保证模型性能. 除此之外，具有脉冲干扰的微分方程能准确的呈现出系统的瞬时变化规律，因此，脉冲随机微分方程吸引了很多学者的关注，详见文

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献（Sakthivel et al.,2013;Ren et al.,2014;Liu et al.,2020）.

另一方面，平均原理作为一种高效、准确的近似分析方法，在非线性动力系统的研究中发挥着重要作用. 它的主要思想是对原始动力系统进行简化得到一个平均系统，并且这个简化后的平均系统可以反映原系统的动力学行为. 目前为止，随机微分系统的平均原理理论已经获得了极大的发展. 例如，Cerrai et al.（2009）研究了一类随机反应扩散模型的平均原理；Ma et al.（2019）研究了Lévy噪声驱动的脉冲随机微分方程的周期平均原理；Cui et al.（2020）在非Lipschitz系数条件下，考虑了脉冲中立型随机微分方程的平均原理；Ahmed et al.（2021）探索出含泊松跳和时滞的Hilfer分数阶随机微分方程的平均原理；Liu et al.（2022a）在非Lipschitz系数条件和无周期条件下，考虑了由分数布朗运动驱动的脉冲随机微分方程的平均原理.

但现有研究存在两方面不足：一是大多数平均原理建立在有限维空间上，很少考虑空间是无穷维的情形（Xu et al.,2020;Liu et al.,2022b），二是Caputo分数阶脉冲随机微分方程已有相应的平均原理研究（Wang et al.,2020;Xu et al.,2011;刘健康等,2023），但Hilfer分数阶脉冲随机发展方程的平均原理尚未见到研究结果. 基于上述讨论，本文在Hilbert空间上考虑如下Hilfer分数阶脉冲随机发展方程的平均原理

$$\begin{cases} D_{0^+}^{\gamma,\beta}x(t) = Ax(t) + f(t,x_t) + h(t,x_t)\dfrac{\mathrm{d}B_Q^H(t)}{\mathrm{d}t}, & t \neq t_k,\ t \in J = (0,b],\\ \Delta x(t_k) = I_k(x(t_k)) = x(t_k^+) - x(t_k^-), & t = t_k, k = 1,2,\cdots,m,\\ x(t) = \varphi(t), & -\lambda \leqslant t < 0,\\ I_{0^+}^{(1-\beta)(1-\gamma)}x(0) = \varphi_0, \end{cases} \tag{1}$$

其中$D^{\gamma,\beta}$是Hilfer分数阶导数，$\gamma \in [0,1]$，$\beta \in \left(\frac{1}{2},1\right)$，$x(\cdot)$取值于实可分Hilbert空间$X$. 闭线性算子$A$：$D(A) \subset X \to X$是强连续算子半群$\{S(t)\}_{t \geqslant 0}$的无穷小生成元. $B_Q^H(t)$是定义在实可分Hilbert空间Y上的分数布朗运动，其中Hurst参数$H \in \left(\frac{1}{2},1\right)$. $\mathcal{PC}([-\lambda,0];X)$指从$[-\lambda,0]$到$X$上所有具有càdlàg路径的连续函数$\varphi$构成的空间，其范数$\|\varphi\|_{\mathcal{PC}} = \sup\limits_{-\lambda \leqslant t \leqslant 0}\|\varphi(t)\| < +\infty$，$x_t = x(t+\tau)(\tau \in [-\lambda,0])$是$\mathcal{PC}$-值的随机过程. $x(t_k^-)$和$x(t_k^+)$分别表示$x(t)$在$t = t_k$时的左极限和右极限，I_k表示$x(t)$在$t = t_k$时刻的脉冲扰动，脉冲时间序列$\{t_k\}$满足$0 < t_1 < \cdots < t_m < t_{m+1} = b$. 系数函数$f$：$J \times \mathcal{PC} \to X$，$h$: $J \times \mathcal{PC} \to \mathcal{L}_2^0(Y,X)$.

1 预备知识

假设$(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t \geqslant 0},P)$是一个带流的完备概率空间，其中$\{\mathcal{F}_t\}_{t \geqslant 0}$满足通常条件，即$\{\mathcal{F}_t\}_{t \geqslant 0}$是右连续的且$\mathcal{F}_0$包含所有零测集. $\{B^H(t)\}_{t \in \mathbb{R}}$是带有Hurst参数$H \in \left(\frac{1}{2},1\right)$的一维分数布朗运动，即$B^H(t)$是一个中心高斯过程且具有以下协方差函数

$$R_H(t,s) = E(B^H(t)B^H(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t,s \in \mathbb{R} = (-\infty,+\infty).$$

记X和Y是两个实可分Hilbert空间，$\mathcal{L}(Y,X)$是从Y映射到X上所有有界线性算子构成的空间. $Q \in \mathcal{L}(Y)$是一个非负自伴算子，满足$Qe_n = \lambda_n e_n$，有限迹$\operatorname{tr} Q = \sum\limits_{n=1}^{\infty}\lambda_n < +\infty$，其中$\{\lambda_n\} \geqslant 0$，$(n = 1,2,\cdots)$是一个非负有界实数序列，$\{e_n\}(n = 1,2,\cdots)$是空间$Y$上一组标准正交基. $\{B_n^H(t)\}_{n \in \mathbb{N}^+}$是独立于完备概率空间$(\Omega,\mathcal{F},P)$的一维标准分数布朗运动序列，现在我们在空间$Y$上定义无穷维分数布朗运动如下：

$$B_Q^H(t) = \sum_{n=1}^{\infty}B_n^H(t)Q^{\frac{1}{2}}e_n = \sum_{n=1}^{\infty}B_n^H(t)\sqrt{\lambda_n}\,e_n, \quad t \geqslant 0,$$

则$B_Q^H(t) \in \mathcal{L}^2(\Omega,Y)$且在空间$Y$中收敛，其中$\mathcal{L}^2(\Omega,Y)$表示所有强可测，平方可积的$Y$-值随机过程组成的

空间.

若$\psi \in \mathcal{L}(Y,X)$并且使得$\psi Q^{\frac{1}{2}}$是Hilbert-Schmidt算子，满足范数$\|\psi\|_{\mathcal{L}_2^0}^2 = \sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\,\psi e_n\right\|^2 < +\infty$，则$\psi$被称为从$Y$映射到$X$的$Q$-Hilbert-Schmidt算子. 记$\mathcal{L}_2^0 \coloneqq \mathcal{L}_2^0(Y,X)$是所有$Q$-Hilbert-Schmidt算子$\psi \in \mathcal{L}(Y,X)$构成的空间，定义空间$\mathcal{L}_2^0$的内积为$\langle \psi_1,\ \psi_2\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty}\langle \psi_1 e_n,\ \psi_2 e_n\rangle$，则$\mathcal{L}_2^0(Y,X)$是一个可分Hilbert空间.

引理1(Abouagwa et al.,2021)　对任意$\phi: J \to \mathcal{L}_2^0(Y,X)$, $\int_0^b \|\phi(s)\|_{\mathcal{L}_2^0}^2 \mathrm{d}s < +\infty$成立, 当$t \in J$, $\sum_{n=1}^{\infty}\left\|\phi(t)Q^{\frac{1}{2}}e_n\right\|$一致收敛，则对任意$t_1$，$t_2 \in J$且$t_2 > t_1$，有

$$E\left\|\int_{t_1}^{t_2}\phi(s)\,\mathrm{d}B_Q^H(s)\right\|^2 \leqslant 2H(t_2 - t_1)^{2H-1}\int_{t_1}^{t_2}\|\phi(s)\|_{\mathcal{L}_2^0}^2 \mathrm{d}s.$$

定义1(Yang et al.,2017a)　函数$f:[a,\ +\infty) \to \mathbb{R}$是一个Lebesgue可积函数，对任意$\beta \in (0,1)$，函数$f$的$\beta$阶Riemann-Liouville积分定义为

$$I_{a^+}^{\beta}f(t) = \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f(s)\mathrm{d}s, \quad t > a, \beta > 0,$$

其中$\Gamma(\cdot)$是Gamma函数.

定义2(Yang et al.,2017a)　函数$f:[a,\ +\infty) \to \mathbb{R}$的$\beta$阶Riemann-Liouville分数阶导数定义为

$${}^{L}D_{a^+}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-s)^{n-1-\beta}f(s)\mathrm{d}s, \quad t > a,\ n-1 < \beta < n,$$

其中$n \in \mathbb{N}^+$.

定义3(Yang et al.,2017a)　函数$f:[a,\ +\infty) \to \mathbb{R}$且$f \in C^n[a,\ +\infty)$，$f$的$\beta$阶Caputo分数阶导数定义为

$${}^{C}D_{a^+}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t (t-s)^{n-1-\beta}f^{(n)}(s)\mathrm{d}s, \quad t > a,\ n-1 < \beta < n,$$

其中$C^n[a,\ +\infty)$表示在区间$[a,\ +\infty)$上n次连续可微的函数构成的空间，$n \in \mathbb{N}^+$.

定义4(Sheng et al.,2022)　函数$f:[a,\ +\infty) \to \mathbb{R}$的Hilfer分数阶导数定义为

$$D_{a^+}^{\gamma,\beta}f(t) = I_{a^+}^{\gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}I_{a^+}^{(1-\gamma)(1-\beta)}f(t), \quad 0 \leqslant \gamma \leqslant 1, 0 < \beta < 1.$$

注1(Sheng et al.,2022)　当$\gamma = 0$，$0 < \beta < 1$，$a = 0$，则Hilfer分数阶导数对应经典的Riemann-Liouville分数阶导数

$$D_{0^+}^{0,\beta}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I_{0^+}^{1-\beta}f(t) = {}^{L}D_{0^+}^{\beta}f(t).$$

当$\gamma = 1$，$0 < \beta < 1$，$a = 0$，则Hilfer分数阶导数对应经典的Caputo分数阶导数

$$D_{0^+}^{1,\beta}f(t) = I_{0^+}^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = {}^{C}D_{0^+}^{\beta}f(t).$$

引理2　方程(1)等价于如下的积分方程

$$\begin{aligned} x(t) \ &= \frac{\varphi_0 t^{(\gamma-1)(1-\beta)}}{\Gamma(\gamma(1-\beta)+\beta)} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(Ax(s) + f(s,x_s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s,x_s)\mathrm{d}B_Q^H(s) + \frac{t^{(\gamma-1)(1-\beta)}}{\Gamma(\gamma(1-\beta)+\beta)}\sum_{0<t_k<t} I_k(x_{t_k}). \end{aligned} \tag{2}$$

证明　可参考文献(Yang et al.,2017a;Ahmed et al.,2018).

为了给出方程(1)的适度解，引入以下Wright-type函数

$$M_{\beta}(\theta) = \sum_{n=1}^{\infty}\frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\beta n)}, \quad 0 < \beta < 1, \theta \in \mathbb{C}.$$

引理3（Yang et al.，2017a） 若积分等式(2)成立，其等价于如下的等式：

$$
\begin{aligned}
x(t) &= S_{\gamma,\beta}(t)\varphi_0 + \int_0^t T_\beta(t-s)f(s,x_s)\mathrm{d}s + \int_0^t T_\beta(t-s)h(s,x_s)\mathrm{d}B_Q^H(s) + \sum_{0<t_k<t} S_{\gamma,\beta}(t-t_k)I_k(x_{t_k}) \\
&= S_{\gamma,\beta}(t)\varphi_0 + \int_0^t (t-s)^{\beta-1}P_\beta(t-s)f(s,x_s)\mathrm{d}s + \int_0^t (t-s)^{\beta-1}P_\beta(t-s)h(s,x_s)\mathrm{d}B_Q^H(s) \\
&\quad + \sum_{0<t_k<t} S_{\gamma,\beta}(t-t_k)I_k(x_{t_k}),
\end{aligned} \tag{3}
$$

其中$P_\beta(t)=\int_0^\infty \beta\theta M_\beta(\theta)S(t^\beta\theta)\mathrm{d}\theta$，$T_\beta(t)=t^{\beta-1}P_\beta(t)$，$S_{\gamma,\beta}(t)=I_{0^+}^{\gamma(1-\beta)}T_\beta(t)$.

定义5 若一个PC-值的随机过程$x:[-\lambda,b]\to X$满足以下条件，则称$x(t)$是方程(1)的适度解.

(i) $x(t)$是$\mathcal{F}_t$-适应的且$\int_0^b E\|x(s)\|^2\mathrm{d}s<+\infty$几乎必然成立；

(ii) $x(t)=\varphi(t)$，$-\lambda\leqslant t\leqslant 0$；

(iii) 当$t\in J$时，$x(t)$具有càdlàg路径且对任意$t\in J$有

$$
\begin{aligned}
x(t) &= S_{\gamma,\beta}(t)\varphi_0 + \int_0^t (t-s)^{\beta-1}P_\beta(t-s)f(s,x_s)\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\beta-1}P_\beta(t-s)h(s,x_s)\mathrm{d}B_Q^H(s) + \sum_{0<t_k<t} S_{\gamma,\beta}(t-t_k)I_k(x_{t_k}).
\end{aligned} \tag{4}
$$

本文中，我们假设如下条件成立：

(H0) 当$t\geqslant 0$时，$S(t)$是一致算子拓扑连续的，且$S(t)$是一致有界的，即存在$M>1$，使得$\sup_{t\in[0,+\infty)}\|S(t)\|<M$.

引理4（Yang et al.，2017b） 在条件(H0)下，对任意$t>0$，$\{P_\beta(t)\}_{t>0}$和$\{S_{\gamma,\beta}(t)\}_{t>0}$是线性算子，且对任意$x\in X$有

$$
\|P_\beta(t)x\|\leqslant\frac{M}{\Gamma(\beta)}\|x\|,\ \|S_{\gamma,\beta}(t)x\|\leqslant\frac{Mt^{(\gamma-1)(1-\beta)}}{\Gamma(\gamma(1-\beta)+\beta)}\|x\|.
$$

定义6（Liu，2007） 设$X_n(n\geqslant 1)$，X是同一概率空间$(\Omega,\mathcal{F},P)$上的随机变量，若$E(|X_n|^2)<+\infty$，且

$$
\lim_{n\to\infty}E(|X_n-X|^2)=0
$$

成立，则称X_n均方收敛于X.

2 平均原理

接下来，我们建立Hilfer分数阶脉冲随机发展方程的平均原理.

首先，定义方程(1)的扰动形式为

$$
\begin{cases}
D_{0^+}^{\gamma,\beta}x_\varepsilon(t)=Ax_\varepsilon(t)+\varepsilon f(t,x_{\varepsilon,t})+\varepsilon^H h(t,x_{\varepsilon,t})\dfrac{\mathrm{d}B_Q^H(t)}{\mathrm{d}t}, & t\neq t_k, t\in J=(0,b],\\
\Delta x_\varepsilon(t_k)=x_\varepsilon(t_k^+)-x_\varepsilon(t_k^-)=\varepsilon I_k(x_\varepsilon(t_k)), & t=t_k, k=1,2,\cdots,m,\\
x_\varepsilon(t)=\varphi(t), & -\lambda\leqslant t<0,\\
I_{0^+}^{(1-\beta)(1-\gamma)}x_\varepsilon(0)=\varphi_0.
\end{cases} \tag{5}
$$

然后根据方程(1)适度解的定义，可以得到方程(5)的适度解为：

$$
\begin{aligned}
x_\varepsilon(t) &= S_{\gamma,\beta}(t)\varphi_0 + \varepsilon\int_0^t (t-s)^{\beta-1}P_\beta(t-s)f(s,x_{\varepsilon,s})\mathrm{d}s \\
&\quad + \varepsilon^H\int_0^t (t-s)^{\beta-1}P_\beta(t-s)h(s,x_{\varepsilon,s})\mathrm{d}B_Q^H(s) + \varepsilon\sum_{0<t_k<t} S_{\gamma,\beta}(t-t_k)I_k(x_{\varepsilon,t_k}),
\end{aligned} \tag{6}
$$

其中$\varepsilon\in(0,\varepsilon_0]$是一个很小的正参数，$\varepsilon_0$是一个固定的常数.

为了得出本文的主要结果，假设系数函数f，h具有周期T，则存在正整数$m\in\mathbb{N}^+$，使得$0<t_1<\cdots<t_m<T$，那么对整数$k>m$，有$t_k=t_{k-m}+T$，$I_k=I_{k-m}$. 现引入可测的系数函数$\bar{f}$：$PC\to X$，$\bar{h}$:

$PC\rightarrow \mathcal{L}_2^0(Y,X)$，$\overline{I}_k$：$PC\rightarrow X$，其中$\bar{f}(x)=\frac{1}{T}\int_0^T f(s,\ x)\mathrm{d}s$，$\bar{h}(x)=\frac{1}{T}\int_0^T h(s,\ x)\mathrm{d}s$，$\bar{I}(x)=\frac{1}{T}\sum_{k=1}^m I_k(x)$.

另外，我们做如下假设：

（H1） 对任意x，$y\in PC$，$t\in J$，存在正常数M_1使得

$$\left\|f(t,\ x)-f(t,\ y)\right\|^2\vee\left\|h(t,\ x)-h(t,\ y)\right\|_{\mathcal{L}_2^0}^2\leqslant M_1^2\left\|x-y\right\|^2.$$

（H2） 对任意的x，$y\in PC$，存在正常数c_k和d_k，使脉冲函数I_k满足

$$\left\|I_k(x)\right\|^2\leqslant c_k,\quad \left\|I_k(x)-I_k(y)\right\|^2\leqslant d_k\left\|x-y\right\|^2.$$

（H3） 对所有$T\in J$，$x\in PC$，存在有界函数$\rho_i(T)>0(i=1,\ 2)$使得

$$\frac{1}{T}\int_0^T\left\|f(s,x)-\bar{f}(x)\right\|^2\mathrm{d}s\leqslant\rho_1(T)\left(1+\left\|x\right\|^2\right),$$

$$\frac{1}{T}\int_0^T\left\|h(s,x)-\bar{h}(x)\right\|^2\mathrm{d}s\leqslant\rho_2(T)\left(1+\left\|x\right\|^2\right),$$

其中$\lim_{T\rightarrow\infty}\rho_i(T)=0(i=1,\ 2)$.

则方程(5)对应如下无脉冲项平均系统：

$$\begin{cases}D_{0^+}^{\gamma,\beta}z_\varepsilon(t)=Az_\varepsilon(t)+\varepsilon\bar{f}(z_{\varepsilon,t})+\varepsilon\bar{I}(z_{\varepsilon,t})+\varepsilon^H\bar{h}(z_{\varepsilon,t})\dfrac{\mathrm{d}B_Q^H(t)}{\mathrm{d}t},\ t\in J=(0,b],\\ I_{0^+}^{(1-\beta)(1-\gamma)}z_\varepsilon(0)=\varphi_0,\\ z_\varepsilon(t)=\varphi(t),\qquad -\lambda\leqslant t<0.\end{cases}\tag{7}$$

参考文献(Gu et al.,2015)中引理2.12的证明，可以得到方程(7)的适度解$z_\varepsilon(t)$为

$$\begin{aligned}z_\varepsilon(t)&=S_{\gamma,\beta}(t)\varphi_0+\varepsilon\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\bar{f}(z_{\varepsilon,s})\mathrm{d}s\\&\quad+\varepsilon^H\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\bar{h}(z_{\varepsilon,s})\mathrm{d}B_Q^H(s)+\varepsilon\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\bar{I}(z_{\varepsilon,s})\mathrm{d}s.\end{aligned}\tag{8}$$

定理1 假设条件(H0)~(H3)成立，则当ε趋于零时，方程(5)的适度解$x_\varepsilon(t)$均方收敛于平均方程(7)的适度解$z_\varepsilon(t)$. 即任意给定一个很小的数$\delta>0$，存在$M_0>0$，$\alpha\in(0,1)$以及$\varepsilon_1\in(0,\varepsilon_0]$，使得当$\varepsilon\in(0,\varepsilon_1]$时有

$$E\left(\sup_{t\in[-\lambda,M_0\varepsilon^{-\alpha}]}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right)\leqslant\delta.$$

证明 由式(6)和式(8)，有

$$\begin{aligned}x_\varepsilon(t)-z_\varepsilon(t)&=\varepsilon\int_0^t(t-s)^{\beta-1}P_\beta(t-s)[f(s,x_{\varepsilon,s})-\bar{f}(z_{\varepsilon,s})]\mathrm{d}s\\&\quad+\varepsilon^H\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\left[h(s,x_{\varepsilon,s})-\bar{h}(z_{\varepsilon,s})\right]\mathrm{d}B_Q^H(s)\\&\quad+\varepsilon\left(\sum_{0<t_k<t}S_{\gamma,\beta}(t-t_k)I_k(x_{\varepsilon,t_k})-\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\bar{I}(z_{\varepsilon,s})\mathrm{d}s\right),\end{aligned}\tag{9}$$

从而对任意$\nu\in(0,b]$,利用基本不等式得到

$$\begin{aligned}E\left(\sup_{0<t\leqslant\nu}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right)&\leqslant3\varepsilon^2E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\left[f(s,x_{\varepsilon,s})-\bar{f}(z_{\varepsilon,s})\right]\mathrm{d}s\right\|^2\right)\\&\quad+3\varepsilon^{2H}E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t(t-s)^{\beta-1}P_\beta(t-s)[h(s,x_{\varepsilon,s})-\bar{h}(z_{\varepsilon,s})]\mathrm{d}B_Q^H(s)\right\|^2\right)\\&\quad+3\varepsilon^2E\left(\sup_{0<t\leqslant\nu}\left\|\sum_{0<t_k<t}S_{\gamma,\beta}(t-t_k)I_k(x_{\varepsilon,t_k})-\int_0^t(t-s)^{\beta-1}P_\beta(t-s)\bar{I}(z_{\varepsilon,s})\mathrm{d}s\right\|^2\right)\\&\leqslant N_1+N_2+N_3.\end{aligned}\tag{10}$$

对于第1项，由引理4可得

$$N_1 \leqslant \frac{6M^2}{\Gamma^2(\beta)}\varepsilon^2 E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t (t-s)^{\beta-1}\left[f(s,x_{\varepsilon,s})-f(s,z_{\varepsilon,s})\right]\mathrm{d}s\right\|^2\right)$$
$$+\frac{6M^2}{\Gamma^2(\beta)}\varepsilon^2 E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t (t-s)^{\beta-1}\left[f(s,z_{\varepsilon,s})-\bar{f}(z_{\varepsilon,s})\right]\mathrm{d}s\right\|^2\right) := N_{11}+N_{12}. \tag{11}$$

利用假设条件（H1）和Cauchy-Schwarz不等式得到

$$N_{11} \leqslant \frac{6M^2M_1^2\varepsilon^2\nu^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}\int_0^\nu E\left(\sup_{0<s_1\leqslant s}\left\|x_{\varepsilon,s_1}-z_{\varepsilon,s_1}\right\|^2\right)\mathrm{d}s = \Lambda_{11}\varepsilon^2\nu^{2\beta-1}\int_0^\nu E\left(\sup_{0<s_1\leqslant s}\left\|x_{\varepsilon,s_1}-z_{\varepsilon,s_1}\right\|^2\right)\mathrm{d}s, \tag{12}$$

其中$\Lambda_{11}=\dfrac{6M^2M_1^2}{(2\beta-1)\Gamma^2(\beta)}$.

由假设条件（H3）得到

$$N_{12} \leqslant \frac{6M^2\varepsilon^2\nu^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}E\left(\sup_{0<t\leqslant\nu} t\cdot\frac{1}{t}\int_0^t\left\|f(s,z_{\varepsilon,s})-\bar{f}(z_{\varepsilon,s})\right\|^2\mathrm{d}s\right) \leqslant \Lambda_{12}\varepsilon^2\nu^{2\beta}, \tag{13}$$

其中$\Lambda_{12}=\dfrac{6M^2}{(2\beta-1)\Gamma^2(\beta)}\sup\limits_{0<t\leqslant\nu}\rho_1(t)\left(1+E\left(\sup\limits_{0<t\leqslant\nu}\left\|z_{\varepsilon,t}\right\|^2\right)\right)$.

对于第2项，由引理4可以推出

$$N_2 \leqslant \frac{6M^2}{\Gamma^2(\beta)}\varepsilon^{2H}E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t (t-s)^{\beta-1}\left[h(s,x_{\varepsilon,s})-h(s,z_{\varepsilon,s})\right]\mathrm{d}B_Q^H(s)\right\|^2\right)$$
$$+\frac{6M^2}{\Gamma^2(\beta)}\varepsilon^{2H}E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t (t-s)^{\beta-1}\left[h(s,z_{\varepsilon,s})-\bar{h}(z_{\varepsilon,s})\right]\mathrm{d}B_Q^H(s)\right\|^2\right) := N_{21}+N_{22}. \tag{14}$$

由引理1、假设条件（H1）和Cauchy-Schwarz不等式得到

$$\begin{aligned} N_{21} &\leqslant \frac{12M^2H}{\Gamma^2(\beta)}\varepsilon^{2H}\nu^{2H-1}E\left(\sup_{0<t\leqslant\nu}\int_0^t (t-s)^{2(\beta-1)}\left\|h(s,x_{\varepsilon,s})-h(s,z_{\varepsilon,s})\right\|^2\mathrm{d}s\right) \\ &\leqslant \Lambda_{21}\varepsilon^{2H}\nu^{2(H+\beta-1)}\int_0^\nu E\left(\sup_{0<s_1\leqslant s}\left\|x_{\varepsilon,s_1}-z_{\varepsilon,s_1}\right\|^2\right)\mathrm{d}s, \end{aligned} \tag{15}$$

其中$\Lambda_{21}=\dfrac{12M^2M_1^2H}{(2\beta-1)\Gamma^2(\beta)}$.

由引理1、假设条件（H1）和假设条件（H3）得到

$$N_{22} \leqslant \frac{12M^2H}{(2\beta-1)\Gamma^2(\beta)}\varepsilon^{2H}\nu^{2(H+\beta-1)}E\left(\sup_{0<t\leqslant\nu} t\cdot\frac{1}{t}\int_0^t\left\|h(s,z_{\varepsilon,s})-\bar{h}(z_{\varepsilon,s})\right\|^2\mathrm{d}s\right) \leqslant \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1}, \tag{16}$$

其中$\Lambda_{22}=\dfrac{12M^2H}{(2\beta-1)\Gamma^2(\beta)}\sup\limits_{0<t\leqslant\nu}\rho_2(t)\left(1+E\left(\sup\limits_{0<t\leqslant\nu}\left\|z_{\varepsilon,t}\right\|^2\right)\right)$.

对于第3项，由基本不等式得到

$$N_3 \leqslant 6\varepsilon^2E\left(\sup_{0<t\leqslant\nu}\left\|\sum_{0<t_k<t}S_{\gamma,\beta}(t-t_k)I_k(x_{\varepsilon,t_k})\right\|^2\right)$$
$$+6\varepsilon^2E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t (t-s)^{\beta-1}P_\beta(t-s)\bar{I}(z_{\varepsilon,s})\mathrm{d}s\right\|^2\right) := N_{31}+N_{32}, \tag{17}$$

由引理4、假设条件（H2）和Cauchy-Schwarz不等式得到

$$\begin{aligned} N_{31} &\leqslant \frac{6\varepsilon^2M^2\nu^{2(\gamma-1)(1-\beta)}}{\Gamma^2(\gamma(1-\beta)+\beta)}E\left(\sup_{0<t\leqslant\nu}\left\|\sum_{0<t_k<t}I_k(x_{\varepsilon,t_k})\right\|^2\right) \leqslant \frac{6\varepsilon^2M^2m\nu^{2(\gamma-1)(1-\beta)}}{\Gamma^2(\gamma(1-\beta)+\beta)}E\left(\sup_{0<t\leqslant\nu}\sum_{k=1}^m\left\|I_k(x_{\varepsilon,t_k})\right\|^2\right) \\ &\leqslant \frac{6\varepsilon^2M^2m^2c_k\nu^{2(\gamma-1)(1-\beta)}}{\Gamma^2(\gamma(1-\beta)+\beta)} = \Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)}, \end{aligned} \tag{18}$$

其中$\Lambda_{31}=\dfrac{6m^2M^2c_k}{\Gamma^2(\gamma(1-\beta)+\beta)}$.

$$N_{32} \leqslant \frac{6M^2\varepsilon^2}{\Gamma^2(\beta)}E\left(\sup_{0<t\leqslant\nu}\left\|\int_0^t(t-s)^{\beta-1}\bar{I}(z_{\varepsilon,s})\mathrm{d}s\right\|^2\right) \leqslant \frac{6M^2\varepsilon^2\nu^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}E\left(\sup_{0<t\leqslant\nu}\int_0^t\left\|\bar{I}(z_{\varepsilon,s})\right\|^2\mathrm{d}s\right)$$

$$\leqslant \frac{6M^2m\varepsilon^2\nu^{2\beta-1}}{(2\beta-1)T^2\Gamma^2(\beta)}E\left(\sup_{0<t\leqslant\nu}\sum_{k=1}^m\int_0^t\left\|I_k(z_{\varepsilon,s})\right\|^2\mathrm{d}s\right) \leqslant \frac{6M^2m^2c_k\varepsilon^2\nu^{2\beta-2}}{(2\beta-1)\Gamma^2(\beta)} = \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)}, \tag{19}$$

其中$\Lambda_{32} = \dfrac{6M^2m^2c_k}{(2\beta-1)\Gamma^2(\beta)}$.

将估计式(11)~(19)代入式(10)，则对任意$\nu \in (0,b]$，得到不等式

$$E\left(\sup_{0<t\leqslant\nu}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right) \leqslant \Lambda_{12}\varepsilon^2\nu^{2\beta} + \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1} + \Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)} + \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)}$$
$$+(\Lambda_{11}\varepsilon^2\nu^{2\beta-1} + \Lambda_{21}\varepsilon^{2H}\nu^{2(H+\beta-1)})\int_0^\nu E\left(\sup_{0<s_1\leqslant s}\left\|x_{\varepsilon,s_1}-z_{\varepsilon,s_1}\right\|^2\right)\mathrm{d}s. \tag{20}$$

令$\Xi(\nu) = E\left(\sup_{0<t\leqslant\nu}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right)$，由于$E\left(\sup_{-\lambda\leqslant t<0}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right)=0$，则

$$\Xi(s+\tau) = E\left(\sup_{0<s_1\leqslant s}\left\|x_{\varepsilon,s_1}-z_{\varepsilon,s_1}\right\|^2\right) = E\left(\sup_{0<s_1\leqslant s}\left\|x_\varepsilon(s_1+\tau)-z_\varepsilon(s_1+\tau)\right\|^2\right), \qquad \tau\in[-\lambda,0),$$

因此，

$$\Xi(\nu) \leqslant \Lambda_{12}\varepsilon^2\nu^{2\beta} + \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1} + \Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)} + \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)}$$
$$+(\Lambda_{11}\varepsilon^2\nu^{2\beta-1} + \Lambda_{21}\varepsilon^{2H}\nu^{2(H+\beta-1)})\int_0^\nu \Xi(s+\tau)\mathrm{d}s. \tag{21}$$

对任意$\nu\in(0,b]$,令$\Theta(\nu) = \sup_{-\lambda\leqslant t\leqslant\nu}\Xi(t)$，则$\Xi(t)\leqslant\Theta(t)$，$\Xi(t+\tau)\leqslant\Theta(t)$，$\tau\in[-\lambda,0)$. 从而得到

$$\Theta(\nu) = \sup_{-\lambda\leqslant t\leqslant\nu}\Xi(t) \leqslant \max\left\{\sup_{-\lambda\leqslant t\leqslant0}\Xi(t) + \sup_{0<t\leqslant\nu}\Xi(t)\right\} \leqslant \Lambda_{12}\varepsilon^2\nu^{2\beta} + \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1}$$
$$+\Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)} + \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)} + \left(\Lambda_{11}\varepsilon^2\nu^{2\beta-1} + \Lambda_{21}\varepsilon^{2H}\nu^{2(H+\beta-1)}\right)\int_0^\nu\Theta(s)\mathrm{d}s. \tag{22}$$

由Gronwall不等式，可推出

$$\Theta(\nu) \leqslant \left(\Lambda_{12}\varepsilon^2\nu^{2\beta} + \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1} + \Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)} + \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)}\right)\exp\left(\Lambda_{11}\varepsilon^2\nu^{2\beta} + \Lambda_{21}\varepsilon^{2H}\nu^{2H+2\beta-1}\right), \tag{23}$$

即有

$$E\left(\sup_{-\lambda<t\leqslant\nu}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right) \leqslant \left(\Lambda_{12}\varepsilon^2\nu^{2\beta} + \Lambda_{22}\varepsilon^{2H}\nu^{2H+2\beta-1} + \Lambda_{31}\varepsilon^2\nu^{2(\gamma-1)(1-\beta)} + \Lambda_{32}\varepsilon^2\nu^{2(\beta-1)}\right)$$
$$\times\exp\left(\Lambda_{11}\varepsilon^2\nu^{2\beta} + \Lambda_{21}\varepsilon^{2H}\nu^{2H+2\beta-1}\right). \tag{24}$$

即存在$M_0>0$和$\alpha\in(0,1)$，使得对所有$t\in(0,M_0\varepsilon^{-\alpha}]\subset(0,b]$满足

$$E\left(\sup_{0<t\leqslant M_0\varepsilon^{-\alpha}}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right) \leqslant \mu\varepsilon^{1-\alpha},$$

其中常数

$$\mu = \left(\Lambda_{12}M_0^{2\beta}\varepsilon^{1+\alpha-2\alpha\beta} + \Lambda_{22}M_0^{2H+2\beta-1}\varepsilon^{2\alpha(1-H-\beta)+2H-1} + \Lambda_{31}M_0^{2(\gamma-1)(1-\beta)}\varepsilon^{2\alpha(1-\gamma)(1-\beta)+\alpha+1}\right.$$
$$\left.+\Lambda_{32}M_0^{2(\beta-1)}\varepsilon^{3\alpha-2\alpha\beta+1}\right)\exp\left(\Lambda_{11}M_0^{2\beta}\varepsilon^{2-2\alpha\beta} + \Lambda_{21}M_0^{2H+2\beta-1}\varepsilon^{2H-\alpha(2H+2\beta-1)}\right). \tag{25}$$

所以对任意给定的数$\delta>0$，存在$\varepsilon_1\in(0,\varepsilon_0]$，使得对任意$\varepsilon\in(0,\varepsilon_1]$和$t\in[-\lambda,M_0\varepsilon^{-\alpha}]\subset[-\lambda,b]$，有

$$E\left(\sup_{t\in[-\lambda,M_0\varepsilon^{-\alpha}]}\left\|x_\varepsilon(t)-z_\varepsilon(t)\right\|^2\right) \leqslant \delta.$$

定理1证毕.

注2 现有文献考虑的是有限维空间上含泊松跳以及Wiener过程的无脉冲扰动的Hilfer分数阶随机微分方程的平均原理(Ahmed et al.,2021;Luo et al.,2021)，与之相比，本文考虑了分数布朗运动驱动的含脉冲项的Hilfer分数阶随机微分方程. 更为重要的是，我们在Hilbert空间上建立了具有算子的Hilfer分数阶

脉冲随机发展方程的平均原理，一定程度上丰富了Hilfer分数阶随机微分方程的平均原理的相关理论.

3 实例

为了说明所得结果的适用性，我们考虑以下含脉冲的Hilfer分数阶随机发展方程

$$
\begin{cases}
D^{\gamma,\frac{2}{3}}x_\varepsilon(t,z)=\dfrac{\partial^2}{\partial z^2}x_\varepsilon(t,z)+\varepsilon\sin^2(t)x_{\varepsilon,t}(z)+2\varepsilon^H\cos^2(t)x_{\varepsilon,t}(z)\dfrac{\mathrm{d}B_Q^H(t)}{\mathrm{d}t},\\
I^{\frac{1}{3}(1-\gamma)}x_\varepsilon(0,z)=\varphi_0,\\
\Delta x_\varepsilon(t_k,z)=\varepsilon\dfrac{x_\varepsilon(t_k,z)}{(4+k)(5+k)},\qquad z\in[0,\pi],k\in1,2,\cdots,m,\\
x_\varepsilon(\theta,z)=\varphi(\theta,z),\qquad \theta\in[-\lambda,0],z\in[0,\pi],\\
x_\varepsilon(t,0)=x_\varepsilon(t,\pi)=0,\qquad t\in(0,m\pi].
\end{cases}
\tag{26}
$$

令空间 $X=Y=L^2[0,\pi]$，系数函数 $f\left(t,x_{\varepsilon,t}\right)=\sin^2(t)x_{\varepsilon,t}(z)$，$h\left(t,x_{\varepsilon,t}\right)=2\cos^2(t)x_{\varepsilon,t}(z)$，脉冲函数 $I_k=\dfrac{x_\varepsilon(t_k,\ z)}{(4+k)(5+k)}$.

定义算子A：$D(A)\to X$，$Ax_\varepsilon(t,z)=\dfrac{\partial^2}{\partial z^2}x_\varepsilon(t,z)$，其中定义域

$$D(A)=\left\{x\in X,\ x,x'\text{全连续},x''\in X,\ x(0)=x(\pi)=0\right\},$$

则A是强连续算子半群$\{S(t)\}_{t\geqslant0}$的无穷小生成元，且对任意$t\geqslant0$，$S(t)$是紧的、解析且自伴的，由一致有界定理可知存在一个常数$M>0$，使得$\|S(t)\|\leqslant M$，且A有离散谱，其特征值是$-n^2$，$n\in\mathbb{N}^+$，对应的标准正交特征向量为$\omega_n(z)=\sqrt{\dfrac{2}{\pi}}\sin(nz),n=1,2,\cdots$,则当$x\in D(A)$时，$Ax=-\sum_{n=1}^{\infty}n^2\langle x,\ \omega_n\rangle\omega_n$.

为了定义算子Q：$Y\to Y$，选择一组非负有界实数序列$\{\lambda_n\}_{n\geqslant1}$，并在$Y$中选取标准正交基$\{e_n\}_{n\geqslant1}$，使得$Qe_n=\lambda_n e_n$成立，并且假设$\mathrm{tr}(Q)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}<+\infty$，从而可以定义随机过程$B_Q^H(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}B_n^H(t)e_n$，其中$H\in(1/2,1)$，$\{B_n^H(t)\}_{n\in\mathbb{N}^+}$，是一个独立于完备概率空间$(\Omega,\mathcal{F},P)$的一维标准分数布朗运动序列.

取$T=\pi$，则

$$\bar f(x)=\frac{1}{\pi}\int_0^\pi f(s,x)\mathrm{d}s=\frac{1}{2}x,\qquad \bar h(x)=\frac{1}{\pi}\int_0^\pi h(s,x)\mathrm{d}s=x,$$

$$\bar I(x)=\frac{1}{\pi}\sum_{k=1}^{m}\frac{x}{(4+k)(5+k)}=\frac{mx}{5(5+m)\pi},$$

于是方程(26)的平均系统为

$$
\begin{cases}
D^{\gamma,\frac{2}{3}}y_\varepsilon(t,z)=Ay_\varepsilon(t,z)+\varepsilon\left(\dfrac{m}{5(5+m)\pi}+\dfrac{1}{2}\right)y_{\varepsilon,t}(z)+y_{\varepsilon,t}\varepsilon^H(z)\dfrac{\mathrm{d}B_Q^H(t)}{\mathrm{d}t},\\
I^{\frac{1}{3}(1-\gamma)}y_\varepsilon(0,z)=\varphi_0,\\
y_\varepsilon(\theta,z)=\varphi(\theta,z),\qquad \theta\in[-\lambda,0],z\in[0,\pi],\\
y_\varepsilon(t,0)=y_\varepsilon(t,\pi)=0.
\end{cases}
\tag{27}
$$

显然，平均系统(27)比原系统(26)简单.假设条件(H0)~(H3)满足，根据定理1，当ε趋于零时，系统(26)的适度解均方收敛于平均系统(27)的适度解.

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